discussing the relative sensitivities of the eigenvalues, the characteristic coefficients, and the traces. The method based on assigning the eigenvalues is found the most worthy of recommendation for reasons of sensitivity and possible numerical overflow. This conclusion strongly corroborates our experience based on using these methods on various examples.

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Efficient Modal Analysis of Damped Large Space Structures

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Introduction

ATTITUDE and shape control of large space structures (LSS) is a problem made extremely difficult by the dynamic properties typical of such vehicles. The most important of these are the very low inherent damping of their flexible modes and the fact that they generally have a large number of low, closely spaced natural frequencies.

A very attractive way of simplifying the overall control problem somewhat is to initially increase the structural damping in some manner. This could be achieved passively, either by means of dampers distributed over the structure or by careful selection of the materials used in its construction[1] (this should allow[2] damping up to about 10% of critical, as opposed to about 0.5% for "standard" LSS). Alternatively, active damping could be introduced by applying feedback between linear or angular velocity sensors and force or torque actuators (not necessarily colocated) positioned throughout the structure. Similar application of position feedback can also be very useful for raising the structure's natural frequencies.

The damped natural frequencies and damping ratios produced by any of these schemes are very expensive to compute, being given from the eigenvalues of a matrix of large dimension. This has prompted various authors to consider the comparatively small amount of damping applied as a perturbation to the original system dynamics: low-order approximations should then give reasonably accurate estimates for the actual damped eigenstructure. In particular, Refs. 3 and 4 implement this for velocity feedback only, while Ref. 5 also allows circulatory effects (a special case of position feedback) and Ref. 6 considers a general "low-authority controller" feedback structure. These methods are far more efficient than a full eigenstructure calculation: however, they were not in the main specifically developed with the LSS case in mind, and tend to suffer from accuracy problems when applied to such a structure with its closely spaced undamped natural frequencies.

This Note presents a new eigenstructure perturbation technique valid for general feedback which minimizes such numerical difficulties by making exclusive use of *unitary* transformations[7] throughout. Note that these complex matrices (or their real subclass, the *orthogonal* matrices) are basic to nearly all of the numerically reliable algorithms developed in control theory in recent years.[8,9] A further practical advantage of the new method is that it gives directly the order of error anticipated in its eigenvalue and eigenvector estimates, as opposed to the incomplete and/or somewhat complicated results of Refs. 3-6.

Problem Formulation

Consider the n-mode undamped model

$$M\ddot{q}+Kq=u \tag{1}$$

for the structural dynamics of an LSS, where q is the vector of generalized coordinates, u the vector of generalized applied forces, and M and K the system mass and stiffness matrices, respectively. Let ω_i be the ith natural frequency of this structure and ϕ_i the corresponding natural mode: $\{\phi_i\}$ can then be normalized so that $\Phi=(\phi_1,...,\phi_n)$ satisfies $\Phi^TM\Phi=I$ and $\Phi^TK\Phi=\mathrm{diag}(\omega_i^2)$. Defining the modal amplitude vector $\hat{q}$ by $q=\Phi\hat{q}$ and, similarly, $\hat{u}=\Phi^Tu$, Eq. (1), in modal form, becomes

$$\ddot{\hat{q}}+\mathrm{diag}(\omega_i^2)\hat{q}=\hat{u} \tag{2}$$

Velocity plus position feedback corresponds to an applied force

$$u=-C\dot{q}-Dq \tag{3}$$

We consider general matrices C and D here: in the terminology of Ref. 5, such a C corresponds to an arbitrary combination of damping and gyroscopic terms, while D is the result of general stiffness and circulatory contributions. Note that for the special case of direct velocity feedback (DVFB), of great practical interest due to its guaranteed absence[10] of "spillover" instabilities, we must have $C=C^T\geq 0$, $D=0$, and colocated sensors and actuators.

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In modal coordinates, the control law Eq. (3) becomes

$$\hat{u} = -\hat{C}\dot{\hat{q}} - \hat{D}\hat{q} \tag{4}$$

for $\hat{C} = \Phi^T C \Phi$ and $\hat{D} = \Phi^T D \Phi$ (only those elements of Φ corresponding to damper or sensor and/or actuator positions are needed to evaluate these matrices). The resulting closed-loop dynamics are thus given by

$$\ddot{\hat{q}} + \hat{C}\dot{\hat{q}} + [\mathrm{diag}(\omega_i^2) + \hat{D}]\hat{q} = 0 \tag{5}$$

which can be rewritten in first-order form by defining the $(2n \times 1)$ vector $x = (\dot{\hat{q}}^T, \hat{q}^T)^T$. We then have $\dot{x} = (A+E)x$, where

$$A = \begin{bmatrix} 0 & \mathrm{diag}(-\omega_i^2) \\ I & 0 \end{bmatrix}, \quad E = \begin{bmatrix} -\hat{C} & -\hat{D} \\ 0 & 0 \end{bmatrix} \tag{6}$$

The complex conjugate eigenvalues $\{\bar{\lambda}\}$ of $(A+E)$ define the natural frequencies $\{\bar{\omega}_i\}$ and damping ratios $\{\zeta_i\}$ of the damped system: $\{\bar{\omega}_i\} = \{\mathrm{Im}(\bar{\lambda}): \mathrm{Im}(\bar{\lambda}) \geq 0\}$ and $\{\zeta_i = \sin\theta: \theta = a\ \tan(-\mathrm{Re}(\bar{\lambda})/\bar{\omega}_i\}$ [for $\zeta_i \ll 1$, this simplifies to $\mathrm{Re}(\bar{\lambda}) \simeq -\zeta_i \bar{\omega}_i$].

Eigenstructure Estimation

The matrix E can be regarded as a perturbation to the open loop A resulting from the applied feedback. If this is chosen to introduce only moderate amounts of damping into the structure, low-order approximation methods can be used to obtain fairly accurate estimates for the actual damped eigenvalues and eigenvectors of $(A+E)$. This has been recognized by various authors, and implemented for certain classes of feedback; in particular, Refs. 3 and 4 consider a general C and $D = 0$ (generalized to the circulatory case $D = -D^T$ in Ref. 5), while Ref. 6 allows a general C and D. However, it will be shown that all of these methods tend to suffer from numerical instability when applied to a generic LSS with its closely spaced natural frequencies. Therefore, their results may contain large errors in this extremely important case.

By contrast, the approximation technique introduced here (for general C and D) is based on the numerically reliable methods used in the eigenstructure sensitivity studies of Ref. 7. The fundamental result is as follows. Let λ be a distinct eigenvalue of A with (right) eigenvector x and left eigenvector y (the solution of $y^H A = \lambda y^H$, where H denotes the conjugate transpose), with these vectors normalized so that $x^H x = y^H x = 1$. Choose the $(2n \times (2n-1)\ U)$ so that (x, U) is unitary [i.e., so that $(x,U)^H(x,U) = I$], and define f, g, h, and $\bar{A}$ by

$$f = U^H E^H x, \quad g = U^H E x, \quad \text{and} \quad (x,U)^H A (x,U) = \begin{pmatrix} \lambda & h^H \\ 0 & \bar{A} \end{pmatrix}$$

[Since (x,U) is unitary it is perfectly conditioned,[7] so no numerical problems arise when applying this similarity transformation.] Then we have the following.

Theorem. The perturbed $(A+E)$ has an eigenvalue $\bar{\lambda}$ and corresponding eigenvector $\bar{x}$, where

$$\bar{\lambda} = \lambda + y^H E x + f^H(\lambda I - \bar{A})^{-1} g + \eta \mathcal{O}(\epsilon^2) + \mathcal{O}(\epsilon^3) \tag{7}$$

and

$$\bar{x} = x + U(\lambda I - \bar{A})^{-1} g + \eta \mathcal{O}(\epsilon^2) \tag{8}$$

provided that the condition

$$\|g\|_2 \frac{(\eta + \epsilon)}{(\delta - \epsilon)^2} < \frac{1}{4} \tag{9}$$

is satisfied, where $\epsilon = \|E\|_2$, $\eta = \|h\|_2$, and $\delta = \|(\lambda I - \bar{A})^{-1}\|_2^{-1}$.

Thus, whenever the simple condition Eq. (9) is satisfied, this method explicitly gives the size of error expected in the first-order approximations for $\bar{\lambda}$ and $\bar{x}$ that are obtained by ignoring the terms involving ϵ in Eqs. (7) and (8). For the purpose of evaluating these error bounds, it is of course prohibitively expensive to compute ϵ exactly as $\|E\|_2$, the largest singular value of E. Instead, standard norm theory[7] can be used to replace it with the much less expensive Frobenius norm $\|E\|_F = \sqrt{\Sigma_{ij} e_{ij}^2}$.

The quantities x, y, U, h, and $\bar{A}$ are independent of E, and therefore need be found only once for each λ when applying this theorem to a general A under various perturbations. When A is of the modal form of Eq. (6), the method becomes even simpler. Let $\lambda = j\omega_i$, where $\omega_i > 0$ is a distinct flexible natural frequency of the structure considered. Then $x = (j\omega_i e_i + e_{n+i})/\gamma$ and $y = \gamma(je_i + \omega_i e_{n+i})/2\omega_i$, where e_i is the ith unit vector and $\gamma = \sqrt{(1+\omega_i^2)}$. U is similarly highly structured: if the columns of (x, U) were reordered to make x column $(n+i)$, the resulting matrix would differ from the identity only in the four intersections of rows and columns i and $(n+i)$, given by

$$\frac{1}{\gamma}\begin{pmatrix} 1 & j\omega_i \\ j\omega_i & 1 \end{pmatrix}$$

Then $h = (1-\omega_i^2)e_i$ (so η is just $|1-\omega_i^2|$), and f, g, and $\bar{A}$ are also of simple form. For instance,

$$\begin{aligned} g_k &= -(j\omega_i \hat{c}_{ki} + \hat{d}_{ki})/\gamma, \quad k = 1,\ldots,n, k \neq i \\ &= -(j\omega_i \hat{c}_{ii} + \hat{d}_{ii})/\gamma^2, \quad k = i \\ &= 0, \quad k = n+1,\ldots,2n-1 \end{aligned} \tag{10}$$

It then follows that the first-order eigenvalue perturbation obtained from Eq. (7) is $\delta\lambda = \delta\lambda_1 + \delta\lambda_2$ in this case, where

$$\delta\lambda_1 = y^H E x = \frac{1}{2\omega_i}(-\omega_i \hat{c}_{ii} + j\hat{d}_{ii}) \tag{11}$$

and

$$\begin{aligned} \delta\lambda_2 &= f^H(j\omega_i I - \bar{A})^{-1} g \\ &= \frac{-1}{2\gamma^4}[(1-\omega_i^2)\hat{c}_{ii}\hat{d}_{ii} + j\omega_i(\hat{c}_{ii}^2 + \hat{d}_{ii}^2)] \\ &\quad + \frac{\omega_i}{\gamma^2} \sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{z_{ki}}{(\omega_k^2 - \omega_i^2)} \end{aligned} \tag{12}$$

where

$$z_{ki} = \omega_i(\hat{c}_{ik}\hat{d}_{ki} + \hat{c}_{ki}\hat{d}_{ik}) + j(\omega_i^2 \hat{c}_{ik}\hat{c}_{ki} - \hat{d}_{ik}\hat{d}_{ki})$$

The corresponding first-order eigenvector perturbation, $\delta x = U(j\omega_i I - \bar{A})^{-1} g$, has components

$$\delta x_i = (-\omega_i \hat{c}_{ii} + j\hat{d}_{ii})/2\omega_i\gamma^3, \quad \delta x_{n+i} = j\omega_i \delta x_i$$

and, for $k = 1,\ldots,n, k \neq i$,

$$\delta x_k = j\omega_i \delta x_{n+k}, \delta x_{n+k} = -(j\omega_i \hat{c}_{ki} + \hat{d}_{ki})/\gamma(\omega_k^2 - \omega_i^2)$$

Evaluating each $\delta\lambda$ of interest requires only about $8n$ multiplications, which compares very favorably, particularly for large n, with operation counts of at least $50n^3$ to calculate all of the eigenvalues of $(A+E)$ by the standard QR algorithm,[7,11] or about $12p^2n$ per eigenvalue if this matrix can be considered sparse[11] with half-bandwidth p. These counts do not include the operations required for the once-only calculation of $\hat{C}$ and $\hat{D}$, each of which requires about mn^2

Table 1 Actual and predicted damping ratios

Mode no.	Undamped frequency, Hz	Damping ratios, %		
		Est. 1[a]	Est. 2[b]	Actual
1	0.0293	24.28	20.41	20.17
2	0.0702	5.14	4.73	4.52
3	0.0763	4.67	4.37	4.23
4	0.1172	5.20	5.01	5.03
5	0.1384	5.10	4.32	4.56
6	0.1546	3.36	4.25	4.07
7	0.1854	3.53	3.62	3.63
8	0.1955	2.94	2.79	2.79

[a]Estimated by the refined "second-order" method of Ref. 6. [b]Estimated by Eqs. (11) and (12).

multiplications for the important special case of m independent sensor/actuator pairs [e.g., $C=\text{diag}(c_1,...,c_m, 0,...,0)$]. Note too that Eq. (12) is simplified somewhat in this case (or, indeed, whenever C and D are symmetric), and is simplified considerably in the DVFB case where $D=0$ as well.

It can be seen from Eq. (12) that $\delta\lambda_2$ becomes potentially far more important when ω_k is near ω_i for some $k\neq i$, the generic LSS case. In fact, it can be shown[7] that $|\delta\lambda_2|\leq\epsilon^2/\delta$, and $\delta\leq\min_k|\omega_k-\omega_i|$: for $\omega_k\simeq\omega_i$ this upper bound on $|\delta\lambda_2|$ becomes quite high. On the other hand, $\delta\lambda_1$ tends to dominate only if all undamped natural frequencies are widely separated: it is exactly the eigenvalue perturbation that would be predicted if we assumed $\delta x=0$ (Jacobi[12]; "first-order" methods of Refs. 3-6), which is clearly reasonable only if no two ω_k are nearly equal. But it is precisely in the calculation of $\delta\lambda_2$ that the present method differs from existing techniques. The corresponding expressions in the "second-order" eigenvalue estimates of Refs. 3-6 are, like $\delta\lambda_2$, typically large for the generic LSS case: this reflects the inherent ill-conditioning of the eigenvalues of such a system. In each method, this translates to inverting a matrix that is ill-conditioned (i.e., nearly singular[7]) for $\omega_k\simeq\omega_i$, or equivalently solving an ill-conditioned set of linear equations. In Ref. 6, this matrix is triangular with diagonal elements $j(\omega_k-\omega_i)$, while in Refs. 3-5 it is the matrix of open-loop eigenvectors (known to be ill-conditioned if two eigenvalues are nearly equal[7]). In the new method, the matrix is $(\lambda I-\bar{A})$: the point of this technique is that its use of unitary transformations ensures that no unnecessary additional numerical accuracy problems are introduced. The resulting contrast in accuracy attainable with the new method is illustrated by the example given in the next section.

Finally, it is interesting to investigate whether Eq. (9) is satisfied for the important case of a low-frequency mode ($\omega_i\ll 1$ rad/s) of a generic LSS ($n\gg 1$; $\delta\simeq 0$, as the $\{\omega_k\}$ are closely spaced) under velocity feedback alone ($D=0$). If we assume, for simplicity, that all elements of $\hat{C}$ are of approximately the same size, we have $\epsilon\simeq n\hat{c}_{ii}$, $\|g\|_2\simeq\sqrt{n}\cdot\hat{c}_{ii}\omega_i$; the left-hand side of Eq. (9) is then approximately [as $\eta=|1-\omega_i^2|\simeq 1, \gamma=\sqrt{(1+\omega_i^2)}\simeq 1]\sqrt{n}\hat{c}_{ii}\omega_i(1+n\hat{c}_{ii})/n^2\hat{c}_{ii}^2$. Now, using Eqs. (11) and (12), $\text{Re}(\delta\lambda)$ ($\simeq -\zeta_i\omega_i$ for small ζ_i) $=-\frac{1}{2}\hat{c}_{ii}$ for $D=0$. Then, for typical $\zeta_i=0.1$, $\hat{c}_{ii}=0.2\omega_i$, and the above expression becomes $(\omega_i+5/n)/\sqrt{n}$. This is certainly smaller than 0.25, as required, so the first-order perturbations $\delta\lambda$ and δx are generically accurate here up to errors as given in Eqs. (7) and (8).

Example

The results of the preceding section are now applied to a structure with dynamics representative of those of an LSS, in that it has a large number of initially undamped modes with low, closely spaced natural frequencies. A uniform beam (either a cantilever or free-free beam) will never satisfy these criteria, as its natural frequencies are always widely separated.[13] Instead, we shall consider a nearly square uniform plate of aluminum ($\rho=2.7\times10^3$ kg/m^3, $E=7.0\times10^{10}$ N/m^2, $\nu=0.34$), $30\times28\times0.005$ m and simply-supported along all four edges. This undamped structure has 45 modes of transverse vibration below 1 Hz; the first eight of these frequencies are tabulated below.

Moderate damping and frequency raising can be introduced into this structure by means of four independent colocated sensor/actuator pairs at locations (6,6), (17,12), (17,16), and (6,22) (in meters, with the x axis along the longer side of plate), each with transverse velocity feedback gain $c=120$ Ns/m and position gain $d=15$ N/m. The resulting fundamental frequency is 0.0332 Hz (an increase of 13%), while the damping ratios of the first eight modes (for a 45-mode model) are tabulated below as estimated by 1) the refined "second-order" method of Ref. 6 and 2) Eqs. (11) and (12). (The techniques of Refs. 3-5 are not applicable to the important case of a symmetric position feedback gain D, as considered here). For comparison, the actual damping ratios (obtained using EISPACK[14]) are also tabulated. The improved accuracy of the new estimates in 2) is clear: in particular, the error produced by Ref. 6 for mode 1 is more than 17 times as large as ours, while the total for all modes shown is nearly 7 times as great.

Conclusions

An improved method has been presented for the efficient prediction of the modal dynamics that result when moderate amounts of velocity and/or position feedback are applied to a flexible structure. This new approach is based on the use of numerically reliable unitary transformations, and has been shown to be a particular improvement in the generic large space structure case of closely spaced undamped natural frequencies.

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